

Deformations of the Exterior Algebra of Differential Forms

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Abstract

Let $D : \Omega \rightarrow \Omega$ be a differential operator defined in the exterior algebra Ω of differential forms over the polynomial ring S in n variables. In this work we give conditions for deforming the module structure of Ω over S induced by the differential operator D , in order to make D an S -linear morphism while leaving the \mathbb{C} -vector space structure of Ω unchanged. One can then apply the usual algebraic tools to study differential operators: finding generators of the kernel and image, computing a Hilbert polynomial of these modules, etc.

Taking differential operators arising from a distinguished family of derivations, we are able to classify which of them allow such deformations on Ω . Finally we give examples of differential operators and the deformations that they induce.

1 Introduction

Let $S = \mathbb{C}[x_1, \dots, x_n]$ be the *ring of polynomials* in n variables and $\Omega = \bigoplus_{r \geq 0} \Omega^r$ the *algebra of differential forms* of S over \mathbb{C} , where Ω^r denotes the module of r -differential forms. Since Ω^r has a natural structure of graded S -module, we will decompose it as $\Omega^r = \bigoplus_{b \geq 0} \Omega^r(b)$, where we assign degree $+1$ to each dx_i . We stress the fact that this (second) grading is the one given by the *Lie derivative* with respect to the *radial vector field* $R = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$, i.e., for $\tau \in \Omega^r(b)$ we have

$$\mathcal{L}_R(\tau) = b \tau$$

as we will recall in eq. 3. In general, for a graded module, we will note in parenthesis the homogeneous component of the given degree; in the case of Ω it will always be the one given by the Lie derivative with respect to the radial vector field R .

For fixed q, a , suppose we are given a differential operator of order one $D : \Omega \rightarrow \Omega$, see Definition 5, such that

$$D(\Omega^r(b)) \subset \Omega^{r+q}(b+a)$$

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for every $r, b \geq 0$.

Regarding D as a morphism of \mathbb{C} -vector spaces would probably lead to finding kernel and image of infinite dimension; on the other side, trying to compute these dimensions on each homogeneous component $\Omega^r(b)$ would force to know D in complete detail. Instead, a module structure on Ω attached to D would lead to more precise and computable information; such as a finite set of generators or a Hilbert polynomial associated to its kernel, or image, exposing discrete invariants.

Both sheaves of principal parts, see [3], and complexes of differential operators, see [4], solve problems of linearization for D . In these works the solution is universal (in a categorical sense) and, as part of the construction, the operator D changes as well as the domain. This fact makes the relation between D and its linearized version difficult to track down.

Our approach to the problem of linearization is focused on the multiplicative action of S over Ω , leaving the operator D invariant as well as the \mathbb{C} -vector space structure on Ω .

As a first example, let us consider the usual *exterior differential* $d : \Omega \rightarrow \Omega$ and denote by i_R the *contraction* with the radial vector field R . As we will show later in Section 5.1, for $f \in S(c)$ and $\tau \in \Omega^r(b)$, where the grading is taken with the Lie derivative with respect to the radial vector field R , in this work we introduce the following action

$$f \cdot_d \tau := \frac{b}{b+c} \left(f\tau + \frac{1}{b} df \wedge i_R \tau \right) \quad (1)$$

which makes the exterior differential $d : \Omega \rightarrow \Omega$ a morphism of S -modules (when $b, c = 0$ we adopt the usual multiplication). Even if we take a different approach, this can be readily verified by direct computations.

In Section 3 we first give a formal definition of the deformations of the exterior algebra Ω over S that we propose, Definition 3.1, and prove a general condition for these modules to be finitely generated.

In Section 4 we state our main result, Theorem 4.5, which is a classification of a distinguished class of differential operators that allow a linearized structure as in Definition 3.1.

This classification exploits the decomposition of a differential operator in terms of a linear map plus a derivation which, in turn, can be decomposed as the Lie derivative plus a contraction with respect to vector valued differential forms. We give the details and definitions of this decomposition in Section 2

Finally, in Section 5 we present two examples of differential operators that allow the linearized structures that we defined.

1.1 Geometric Motivation

Even if our approach to the subject is purely algebraic, there is a geometric nature in our work that we would like to state here.

A codimension one foliation in projective space \mathbb{P}^{n-1} of degree $a-2$, $Fol^1(\mathbb{P}^{n-1}, a-2)$, is given by a differential 1-form $\omega \in \Omega^1(a)$ such that it descends to projective space, *i.e.*, $i_R\omega = 0$, and such that verifies the Frobenius integrability condition $\omega \wedge d\omega = 0$. As it is shown in [2], the Zariski tangent space to the space of such foliations can be parameterized by

$$T_\omega Fol^1(\mathbb{P}^{n-1}, a-2) = \{\eta \in \Omega^1(a) : i_R\eta = 0 \text{ and } \omega \wedge d\eta + d\omega \wedge \eta = 0\}.$$

Using the second equation, C. Camacho and A. Lins-Neto, in [1], define the following notion of regularity of an integrable, homogeneous, differential 1-form and prove an associated stability result. By looking at ω as a homogeneous affine form in \mathbb{C}^n , ω is said to be regular if for every $a < e$ the graded complex of homogeneous elements

$$\begin{array}{ccccccc} T(a-e) & \longrightarrow & \Omega^1(a) & \longrightarrow & \Omega^3(a+e) & & (2) \\ X & \longmapsto & \mathcal{L}_X(\omega) & & & & \\ & & \eta & \longmapsto & \omega \triangle \eta := \omega \wedge d\eta + d\omega \wedge \eta & & \end{array}$$

has trivial homology in degree 1, where we denote as $T := (\Omega^1)^*$ to the module of *vector fields* and assign degree -1 to each $\frac{\partial}{\partial x_i} := dx_i^*$.

Studying the function $a \mapsto \varphi_\omega(a) := \dim_{\mathbb{C}}(Ker(\omega \triangle -)(a))$, for every $a \in \mathbb{N}$, it was clear the necessity of a better understanding of the differential operator $\omega \triangle -$ what led us to the present work (among other things, as we will briefly mention in 5.2).

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2 Basic Definitions and General Setting

Along this section we give the definitions of our main objects of study, which are derivations, differential operators and vector valued differential forms on Ω . With these objects we state a decomposition theorem for a differential operator on Ω , see Theorem 2.4. Such decomposition theorem, as far as we know, is missing in the literature.

Let us first recall some basic definitions, see *e.g.* [6]:

A *linear map of degree q* in Ω is a linear map $\varphi : \Omega \rightarrow \Omega$ such that $\varphi(\Omega^r) \subset \Omega^{r+q}$ for every $r \geq 0$.

A *derivation of degree q* in Ω is a linear map $D_0 : \Omega \rightarrow \Omega$ of degree q such that

$$D_0(\mu \wedge \tau) = D_0(\mu) \wedge \tau + (-1)^{r_q} \mu \wedge D_0(\tau)$$

for $\mu \in \Omega^r$ and $\tau \in \Omega^s$. If $D(\Omega^r(b)) \subset \Omega^{q+r}(b+a)$ we will say that D has *bidegree (q, a)* and denote by $\text{Der}^q(\Omega)$ and $\text{Der}^{(q,a)}(\Omega)$ the spaces of derivations of degree q and bidegree (q, a) respectively. In case that D_0 is S -linear we will say that D_0 is a *linear derivation*.

We recall from the introduction that we will write as $T := (\Omega^1)^*$ to the module of vector fields and assign degree -1 to each $\frac{\partial}{\partial x_i} := dx_i^*$.

In Ω we have the exterior differential d which is a derivation of degree 1 and, for $X \in T$, the contraction i_X which is a linear derivation of degree -1 .

Let $\varphi, \psi : \Omega \rightarrow \Omega$ be two linear maps of degree q and r respectively. We define the (*graded*) *Lie bracket* $[\varphi, \psi]$ as

$$[\varphi, \psi] := \varphi \circ \psi - (-1)^{qr} \psi \circ \varphi.$$

Following Cartan's formula, see *e.g.* [8], the *Lie derivative* with respect to X , \mathcal{L}_X , can be computed as

$$\mathcal{L}_X = [i_X, d] = i_X d + d i_X$$

which is a derivation of degree 0.

Taking $\tau \in \Omega^r(b)$, the Lie derivative with respect to the radial vector field R , verifies the well known formula, see *e.g.* [5],

$$\mathcal{L}_R(\tau) = i_R d\tau + d i_R \tau = b \tau \quad (3)$$

which decomposes in a unique way a differential form in a radial plus and exact form.

A linear derivation D_0 of degree q is uniquely determined by its restriction to 1-forms $D_0 : \Omega^1 \rightarrow \Omega^{q+1}$, which can be viewed as an element $L \in \Omega^{q+1} \otimes T$.

For $q \geq -1$, we will say that $L \in \Omega^{q+1} \otimes T$ is a *vector valued differential form* and note Ω_T^{q+1} to the space of such elements. For $\tau \in \Omega^r$, we define the contraction $i_L \tau \in \Omega^{q+r}$ by the formula

$$\begin{aligned} i_L \tau(X_1, \dots, X_{q+r}) &:= \\ &= \frac{1}{(q+1)!(r-1)!} \sum_{\sigma \in S_{q+r}} \text{sign}(\sigma) \tau(L(X_{\sigma(1)}, \dots, X_{\sigma(q+1)}), X_{\sigma(q+2)}, \dots, X_{\sigma(q+r)}) \end{aligned} \quad (4)$$

for $X_1, \dots, X_{q+r} \in T$ and S_{q+r} the permutation group of $q+r$ elements.

We recall from [6] the following two propositions that give a classification of derivations:

Proposition 2.1. *If $L \in \Omega_T^{q+1}$, then $i_L \in \text{Der}^q(\Omega)$ and any linear derivation is of this form.*

Proof. We just notice that we are using the the identification of a differential r -form as an alternating map $(T)^{\otimes r} \rightarrow S$, see e.g. [8]. For the rest, we follow [6, Chapter IV, 16.2, p. 192]. \square

Using eq. 3 and Proposition 2.1 we can define the Lie derivative for a vector valued differential form, $K \in \Omega_T^q$, as

$$\mathcal{L}_K := [i_K, d] \in \text{Der}^q(\Omega).$$

Proposition 2.2. *If $D_0 \in \text{Der}^q(\Omega)$, then there exists unique $K \in \Omega_T^q$ and $L \in \Omega_T^{q+1}$ such that*

$$D_0 = \mathcal{L}_K + i_L.$$

Proof. See [6, Chapter IV, 16.3, p. 193]. \square

For $\tau \in \Omega$ we will denote by λ_τ the endomorphism of *left multiplication* by τ in Ω . It is immediate to see that a linear map $\varphi : \Omega \rightarrow \Omega$ of degree q is Ω -linear, i.e. $\varphi(\mu) = \varphi(1) \wedge \mu$, if and only if

$$[\varphi, \lambda_\tau] = 0$$

for every $\tau \in \Omega$. We then define:

A *differential operator of order 1 and degree q* , or simply a *differential operator of degree q* , in Ω is a linear map $D : \Omega \rightarrow \Omega$ of degree q such that

$$[[D, \lambda_\mu], \lambda_\tau] = 0 \tag{5}$$

for all $\mu, \tau \in \Omega$. If $D(\Omega^r(b)) \subset \Omega^{q+r}(b+a)$ we will say that D has bidegree (q, a) and denote by $\text{Diff}^q(\Omega)$ and $\text{Diff}^{(q,a)}(\Omega)$ the spaces of differential operators of degree q and bidegree (q, a) respectively.

Even if the following proposition is very well known, we add a proof just to show that the sign rule arising from the skew commutativity of Ω does not make any conflicts.

Proposition 2.3. *Let $D \in \text{Diff}^q(\Omega)$. Then D can be decomposed as*

$$D = (D - \lambda_{D(1)}) + \lambda_{D(1)}$$

where $D - \lambda_{D(1)} \in \text{Der}^q(\Omega)$ and $\lambda_{D(1)}$ is a linear map.

Proof. Evaluating at 1 the formula $[[D, \lambda_\mu], \lambda_\tau] = 0$ we get

$$D(\mu \wedge \tau) + D(1) \wedge \mu \wedge \tau = D(\mu) \wedge \tau + (-1)^{qr} \mu \wedge D(\tau).$$

And by subtracting $-2 D(1) \wedge \mu \wedge \tau$ in both sides we see that

$$D(\mu \wedge \tau) - D(1) \wedge \mu \wedge \tau = (D(\mu) - D(1) \wedge \mu) \wedge \tau + (-1)^{qr} \mu \wedge (D(\tau) - D(1) \wedge \tau)$$

showing that $D - \lambda_{D(1)} \in \text{Der}^q(\Omega)$. \square

As a direct corollary of Proposition 2.2 and Proposition 2.3, we have the following decomposition for a differential operator:

Theorem 2.4. *Let $D \in \text{Diff}^q(\Omega)$. Then D can be written as*

$$D = \mathcal{L}_K + i_L + \lambda_\mu$$

for unique $K \in \Omega_T^q$, $L \in \Omega_T^{q+1}$ and $\mu \in \Omega^q$.

3 Deformations of the Exterior Algebra

In Definition 3.1 we give formal definition of the deformations of Ω induced by a differential operator $D \in \text{Diff}^{(q,a)}(\Omega)$ and then, in Proposition 3.2, we give conditions to these modules to be finitely generated.

As eq. 1 shows, these deformations have some denominators in the formula, that can be zero in low degrees. Because of that, we first need a technical definition that will allow us to avoid this situation.

Two graded S -modules M and N are said to be *stably isomorphic* if there exists an $n_0 \in \mathbb{N}$ such that $M(k) \simeq N(k)$ for every $k \geq n_0$.

Let $\tilde{\Omega}$ be an algebra of differential forms stably isomorphic to Ω . Without loss of generality we can assume that

$$\tilde{\Omega} = \bigoplus_{\substack{r \geq 0 \\ b \geq n_r}} \Omega^r(b)$$

for some $n_r \in \mathbb{N}$.

Definition 3.1. *Let $D \in \text{Diff}^{(q,a)}(\tilde{\Omega})$. For $f \in S(c)$ and $\tau \in \Omega^r(b)$ we define the following action of S in $\Omega^r(b)$*

$$f \cdot_D \tau = \alpha f \tau + \beta df \wedge i_R \tau$$

where $\alpha = \alpha(r, b, c)$ and $\beta = \beta(r, b, c)$ verify the conditions

- a) $\alpha(-, -, 0) = 1$
- b) $D(f \cdot_D \tau) = f \cdot_D D(\tau)$
- c) $(gf) \cdot_D \tau = g \cdot_D (f \cdot_D \tau)$.

We will note Ω_D to $\tilde{\Omega}$ under this action from S .

In case such α and β exists, Ω_D gets a structure of a graded S -module extending the usual multiplication from \mathbb{C} and

$$\Omega_D \xrightarrow{D} \Omega_D$$

is S -linear.

It is clear that the modules Ω_D^r are stably free of rank $\binom{n}{r}$. Next we give a condition for Ω_D to be finitely generated.

Proposition 3.2. *Let $\Omega_D^r = \bigoplus_{b \geq n_r} \Omega_D^r(b)$ with $n_r > n$ and let $\alpha(r, b, c)$ and $\beta(r, b, c)$ as in Definition 3.1. If for every $b \geq n_r$ we have*

$$\alpha(r, b, 1)^2 - \beta(r, b, 1)^2 \neq 0$$

then Ω_D^r is finitely generated by elements of degree n_r .

Proof. Let us fix (r) and b and denote $\alpha = \alpha(r, b, 1)$ and $\beta = \beta(r, b, 1)$. Consider the multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0$ such that $\sum_{i=1}^n \gamma_i = g + 1 = b + 1 - r$ and let $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$.

Take $b \geq n_r$. Every element of $\Omega_D^r(b+1)$ it is a sum of elements of the form

$$x^\gamma dx_I$$

where we write $x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}$ and $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_r}$.

Let $k \in \{1, \dots, n\}$ such that $\gamma_k \neq 0$. Then we have the formula

$$x_k \cdot_D (x^{\gamma - e_k} dx_I) = \alpha x^\gamma dx_I + \beta \sum_{j=1}^r x^{\gamma - e_k + e_{i_j}} dx_k \wedge i_{\frac{\partial}{\partial x_{i_j}}} dx_I \quad (6)$$

where we note e_i to the i -th canonical vector.

If $k \in I$, then eq. 6 equals to $(\alpha + \beta) x^\gamma dx_I$. Since $\alpha^2 - \beta^2 \neq 0$ we have

$$\frac{1}{\alpha + \beta} x_k \cdot_D (x^{\gamma - e_k} dx_I) = x^\gamma dx_I \quad (7)$$

when $k \in I$ and $\gamma_k \neq 0$.

On the other side, assume that for all k such that $\gamma_k \neq 0$ we have $k \notin I$. By hypothesis we know $b \geq n_r > n$. Replacing $b > n$ in the equation $b = g + r$ we have $g > n - r$. Then, necessarily there exists k such that $\gamma_k \geq 2$.

Let $\ell \in \{1, \dots, r\}$ such that $\ell \in I$ and k such that $\gamma_k \geq 2$. We have

$$\begin{aligned} x_\ell \cdot_D \left(x^{\gamma - e_k} dx_k \wedge i_{\frac{\partial}{\partial x_\ell}} dx_I \right) &= \\ &= \alpha x^{\gamma - e_k + e_\ell} dx_k \wedge i_{\frac{\partial}{\partial x_\ell}} dx_I + \beta \sum_{t \in I \cup \{k\}} x^{\gamma - e_k + e_t} dx_\ell \wedge i_{\frac{\partial}{\partial x_t}} \left(dx_k \wedge i_{\frac{\partial}{\partial x_\ell}} dx_I \right) = \\ &= \alpha x^{\gamma - e_k + e_\ell} dx_k \wedge i_{\frac{\partial}{\partial x_\ell}} dx_I + \beta x^\gamma dx_I - \beta \sum_{j=1}^r x^{\gamma - e_k + e_{i_j}} dx_k \wedge i_{\frac{\partial}{\partial x_{i_j}}} dx_I. \end{aligned} \quad (8)$$

Operating with eq. 6 and eq. 8 we get

$$\begin{aligned} & \alpha \left[x_k \cdot_D (x^{\gamma-e_k} dx_I) \right] - \beta \left[x_\ell \cdot_D (x^{\gamma-e_k} dx_k \wedge i_{\frac{\partial}{\partial x_\ell}} dx_I) \right] = \\ & = (\alpha^2 - \beta^2) x^\gamma dx_I + (\alpha + \beta) \beta \sum_{j=1}^r x^{\gamma-e_k+e_{i_j}} dx_k \wedge i_{\frac{\partial}{\partial x_{i_j}}} dx_I. \end{aligned} \quad (9)$$

Let us call $J = (I \setminus \{\ell\}) \cup \{k\}$. In each of the terms of the right of eq. 9 we have $k \in J$ and $\gamma_k \geq 2$. Then $(\gamma - e_k + e_{i_j})_k \neq 0$ and we can clear all the right terms of eq. 9 following eq. 7, obtaining $(\alpha^2 - \beta^2)x^\gamma dx_I$.

This way, every element is generated by elements of the previous degree starting from $b+1 \geq n_r$ and the result follows. \square

Remark 3.3. *We would like to point out that we do not know what happens with the converse statement, i.e., in the case where $\alpha(r, b, 1)^2 - \beta(r, b, 1)^2 = 0$.*

Remark 3.4. *It is worth mentioning that $\Omega_D^r \neq (\Omega_D^1)^{\wedge r}$. For this, it is enough to compute the products*

$$(z \cdot_D dx) \wedge dy \quad dx \wedge (z \cdot_D dy)$$

and see that they are different.

4 Classification of Differential Operators

In Theorem 4.5 we classify which differential operators allow a linearization as the one in Definition 3.1 for a distinguished class of differential operators.

There is a distinguished vector valued differential form $Id \in \Omega_T^1$, which is the one arising from the identity map on $\Omega^1 \rightarrow \Omega^1$; then, Id takes the form $Id = \sum_{i=1}^n dx_i \otimes \frac{\partial}{\partial x_i}$. Following eq. 4, we can apply i_{Id} to an r -product of 1-differential forms $\tau = \tau_1 \wedge \dots \wedge \tau_r$ and get the formula

$$i_{Id}(\tau) = r \tau.$$

Then, it is immediate that the usual exterior differential can be computed as

$$\mathcal{L}_{Id}(\tau) = d\tau.$$

The space of derivations arising from the module structure of Ω on the space generated by $(\mathcal{L}_{Id}, i_{Id})$ it is given by $(\mathcal{L}_{\omega_1 \wedge Id}, i_{\omega_2 \wedge Id})_{\{\omega_1, \omega_2 \in \Omega\}}$; this can be easily seen by the following equalities, see [6, Chapter IV, 16.7, Theorem, p. 194],

$$\mathcal{L}_{\omega \wedge Id} = \omega \wedge \mathcal{L}_{Id} + (-1)^q d\omega \wedge i_{Id} \quad \text{and} \quad i_{\omega \wedge Id} = \omega \wedge i_{Id},$$

where $\omega \in \Omega^r(q)$. This allow us to define:

Definition 4.1. We define the space of differential operators of bidegree (q, a) associated to $Id \in \Omega_T^1$ as

$$Diff_{Id}^{(q,a)}(\Omega) = \{ \tilde{\omega}_1 \wedge \mathcal{L}_{Id} + \tilde{\omega}_2 \wedge i_{Id} + \lambda_{\tilde{\mu}} : \text{for some } \tilde{\omega}_1 \in \Omega^{q-1}(a), \tilde{\omega}_2, \tilde{\mu} \in \Omega^q(a) \}$$

Definition 4.2. For $q \geq 1$, we define the set $Lin^{(q,a)}(\Omega)$ of linearizable differential operators of bidegree (q, a)

$$Lin^{(q,a)}(\Omega) = \left\{ \omega_1 \wedge \mathcal{L}_{Id} + \left(\frac{1}{q} d\omega_1 + \omega_2 \right) \wedge i_{Id} + (t \lambda_{d\omega_1} + \lambda_{\mu}) : \text{for some} \right. \\ \left. \omega_1 \in \Omega^{q-1}(a) \text{ and } \omega_2, \mu \in \Omega^q(a) \text{ such that} \right. \\ \left. i_R \omega_1 = i_R \omega_2 = i_R \mu = 0 \text{ and } t \in \mathbb{C} \right\}.$$

Remark 4.3. Notice that $Lin^{(q,a)} \subsetneq Diff^{(q,a)}$. This can be seen by using the decomposition of eq. 3 together with the conditions $i_R \omega_1 = i_R \omega_2 = i_R \mu = 0$, that fixes the exact form of the forms $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\mu}$ to 0, $\frac{1}{q} d\omega_1$, and $td\omega_1$, respectively.

Remark 4.4. One can turn $Lin^{(q,a)}(\Omega)$ in a \mathbb{C} -vector space in the following way: for D and D' in $Lin^{(q,a)}(\Omega)$ defined as

$$D = \omega_1 \wedge \mathcal{L}_{Id} + \left(\frac{1}{q} d\omega_1 + \omega_2 \right) \wedge i_{Id} + (t \lambda_{d\omega_1} + \lambda_{\mu}) \\ D' = \omega'_1 \wedge \mathcal{L}_{Id} + \left(\frac{1}{q} d\omega'_1 + \omega'_2 \right) \wedge i_{Id} + (t' \lambda_{d\omega'_1} + \lambda_{\mu'})$$

we define the addition as

$$D + D' := (\omega_1 + \omega'_1) \wedge \mathcal{L}_{Id} + \left(\frac{1}{q} d(\omega_1 + \omega'_1) + (\omega_2 + \omega'_2) \right) \wedge i_{Id} + \\ + \left[(t + t') \lambda_{d(\omega_1 + \omega'_1)} + \lambda_{(\mu + \mu')} \right].$$

Also, there is no ambiguity in the way these differential forms are written, since $\frac{1}{q} d\omega_1 + \omega_2$ and $t d\omega_1 + \mu$ are the addition of a radial plus an exact term, for which eq. 3 assures uniqueness of writing.

With the following theorem we classify which differential operators arising from Id can be linearized.

Theorem 4.5. Let $D \in Diff_{Id}^{(q,a)}(\tilde{\Omega})$. Then there exists α and β that verify the conditions of Definition 3.1 making D an S -linear operator if and only if $D \in Lin^{(q,a)}(\tilde{\Omega})$, for $q \geq 1$. If $D \in Lin^{(q,a)}(\tilde{\Omega})$ is given by

$$D(\tau) = \omega_1 \wedge \mathcal{L}_{Id} + \left(\frac{1}{q} d\omega_1 + \omega_2 \right) \wedge i_{Id} + (t \lambda_{d\omega_1} + \lambda_{\mu})$$

with $\omega_1 \neq 0$, then α and β can be chosen to be

$$\alpha(r, b, c) := \frac{b - a \left(\frac{r}{q} + (-1)^q t \right)}{b + c - a \left(\frac{r}{q} + (-1)^q t \right)} \quad \beta(r, b, c) := \frac{\alpha(r, b, c)}{b - a \left(\frac{r}{q} + (-1)^q t \right)}$$

When $\omega_1 = 0$, the usual multiplication law can be used.

Proof. Take $D \in \text{Diff}_{Id}^{(q,a)}(\tilde{\Omega})$. Then D can be written as

$$D = \omega_1 \wedge \mathcal{L}_{Id} + \omega_2 \wedge i_L + \lambda_\mu$$

for some $\omega_1 \in \Omega^{q-1}(a)$ and $\omega_2, \mu \in \Omega^q(a)$.

It will be convenient to decompose ω_1, ω_2 and μ in the following way

$$\omega_1 = \omega_{1r} + \omega_{1d} \quad \omega_2 = \omega_{2r} + \omega_{2d} + t_1 d\omega_{1r} \quad \mu = \mu_r + \mu_d + t_2 d\omega_{1r}$$

where the subindex r and d denote radial and exact terms, and ω_{2d} and μ_d are linearly independent to $d\omega_{1r}$. For $\tau \in \Omega^r(b)$, we then have

$$D(\tau) = (\omega_{1r} + \omega_{1d}) \wedge d\tau + \left[(rt_1 + t_2) d\omega_{1r} + (r\omega_{2r} + \mu_r) + (r\omega_{2d} + \mu_d) \right] \wedge \tau \quad (10)$$

$$D(\tau) = (\omega_{1r} + \omega_{1d}) \wedge d\tau + \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge \tau \quad (11)$$

where we are writing $\nu_r = r\omega_{2r} + \mu_r$ and $\nu_d = r\omega_{2d} + \mu_d$.

We now want to see under what conditions we have the equalities

$$\text{b) } f \cdot D(\tau) = D(f \cdot \tau)$$

$$\text{c) } g \cdot (f \cdot \tau) = (gf) \cdot \tau$$

for the action defined in Definition 3.1: $f \cdot \tau = \alpha f \tau + \beta df \wedge i_R \tau$.

From b) we get

$$\begin{aligned} f \cdot D(\tau) &= f \cdot \left\{ (\omega_{1r} + \omega_{1d}) \wedge d\tau + \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge \tau \right\} = \\ &= \alpha(r + q, b + a, c) f \left\{ (\omega_{1r} + \omega_{1d}) \wedge d\tau + \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge \tau \right\} + \\ &\quad + \beta(r + q, b + a, c) df \wedge i_R \left((\omega_{1r} + \omega_{1d}) \wedge d\tau + \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge \tau \right) = \\ &= \alpha(r + q, b + a, c) f \left\{ (\omega_{1r} + \omega_{1d}) \wedge d\tau + \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge \tau \right\} + \\ &\quad + \beta(r + q, b + a, c) df \wedge \left\{ i_R \omega_{1d} \wedge d\tau + (-1)^{q-1} (\omega_{1r} + \omega_{1d}) \wedge i_R d\tau + \right. \\ &\quad \left. + \left[a(rt_1 + t_2) \omega_{1r} + i_R \nu_d \right] \wedge \tau + (-1)^q \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge i_R \tau \right\} \end{aligned} \quad (12)$$

and, writing $\alpha = \alpha(r, b, c)$,

$$\begin{aligned} D(f \cdot \tau) &= D(\alpha f \tau + \beta df \wedge i_R \tau) = \\ &= \alpha (\omega_{1r} + \omega_{1d}) \wedge df \wedge \tau + \alpha f (\omega_{1r} + \omega_{1d}) \wedge d\tau - \beta (\omega_{1r} + \omega_{1d}) \wedge df \wedge di_R \tau + \\ &\quad + \left[(rt_1 + t_2) d\omega_{1r} + \nu_r + \nu_d \right] \wedge (\alpha f \tau + \beta df \wedge i_R \tau). \end{aligned} \quad (13)$$

Taking the coefficients from every different term in eqs. 12 and 13, we get the system

$$\left\{ \begin{array}{ll} \text{I) } f \omega_{1r} \wedge d\tau : & \alpha(r+q, b+a, c) = \alpha(r, b, c) \\ \text{II) } f \omega_{1d} \wedge d\tau : & \alpha(r+q, b+a, c) = \alpha(r, b, c) \\ \text{III) } f d\omega_{1r} \wedge \tau : & \alpha(r+q, b+a, c)(rt_1 + t_2) = \alpha(r, b, c)(rt_1 + t_2) \\ \text{IV) } f \nu_r \wedge \tau : & \alpha(r+q, b+a, c) = \alpha(r, b, c) \\ \text{V) } f \nu_d \wedge \tau : & \alpha(r+q, b+a, c) = \alpha(r, b, c) \\ \text{VI) } df \wedge i_R \omega_{1d} \wedge d\tau : & \beta(r+q, b+a, c) = 0 \\ \text{VII) } df \wedge \omega_{1r} \wedge i_R d\tau : & (-1)^{q-1} \beta(r+q, b+a, c) = (-1)^{q-1} \beta(r, b, c) \\ \text{VIII) } df \wedge \omega_{1d} \wedge i_R d\tau : & (-1)^{q-1} \beta(r+q, b+a, c) = (-1)^{q-1} \beta(r, b, c) \\ \text{IX) } df \wedge \omega_{1r} \wedge \tau : & \beta(r+q, b+a, c)a(rt_1 + t_2) = \\ & = (-1)^{q-1}(\alpha(r, b, c) - b\beta(r, b, c)) \\ \text{X) } df \wedge i_R \nu_d \wedge \tau : & \beta(r+q, b+a, c) = 0 \\ \text{XI) } df \wedge d\omega_{1r} \wedge i_R \tau : & \beta(r+q, b+a, c)(-1)^q(rt_1 + t_2) = \\ & = \beta(r, b, c)(-1)^q(rt_1 + t_2) \\ \text{XII) } df \wedge \nu_r \wedge i_R \tau : & (-1)^q \beta(r+q, b+a, c) = (-1)^q \beta(r, b, c) \\ \text{XIII) } df \wedge \nu_d \wedge i_R \tau : & (-1)^q \beta(r+q, b+a, c) = (-1)^q \beta(r, b, c) \end{array} \right.$$

which can be reduced to

$$\left\{ \begin{array}{ll} \text{XI) } df \wedge d\omega_{1r} \wedge i_R \tau : & (-1)^{q-1} \beta(r+q, b+a, c) = (-1)^{q-1} \beta(r, b, c) \\ \text{IV) } f \nu_r \wedge \tau : & \alpha(r+q, b+a, c) = \alpha(r, b, c) \\ \text{IX) } df \wedge \omega_{1r} \wedge \tau : & \beta(r+q, b+a, c)a(rt_1 + t_2) = \\ & = (-1)^{q-1}(\alpha(r, b, c) - b\beta(r, b, c)) \\ \text{X) } df \wedge i_R \nu_d \wedge \tau : & \beta(r+q, b+a, c) = 0 \end{array} \right.$$

It is clear the sufficiency of these equalities to accomplish b). For the necessity we proceed as follows:

- if $i_R \tau = 0$, then the terms involving eqs. I), ..., X) must coincide, leaving the terms of eqs. XI), XII) and XIII) apart which can be considered in another system. Since these equations are linearly independent by hypothesis, we get that eq. XI) must be satisfied.
- if $\tau = df \wedge \rho_d$, for some exact differential form ρ_d , then only eqs. III), IV) and V) survive. Also they are linearly independent by hypothesis, then eq. IV) must be also satisfied.

- by the previous arguments, we can clear off eqs. I) to V), VII), VIII) and XI), XII), XIII). Taking now $\tau = \tau_d$ we get only eqs. IX) and X) which are also linearly independent.

Using eq. IX) we can clear β as

$$\beta(r, b, c) = \frac{\alpha(r, b, c)}{b - (-1)^q a(r t_1 + t_2)}.$$

From this equality and eqs. I) and VII) we get the formula

$$\frac{\alpha(r, b, c)}{b - (-1)^q a(r t_1 + t_2)} = \frac{\alpha(r, b, c)}{(b + a) - (-1)^q a((r + q)t_1 + t_2)}$$

from where we are able to clear t_1 as $\frac{(-1)^q}{q}$ and obtain the system

$$\begin{cases} \alpha(r, b, c) = \alpha(r + q, b + a, c) \\ \beta(r, b, c) = \frac{\alpha(r, b, c)}{b - a\left(\frac{r}{q} + (-1)^q t_2\right)} \end{cases} \quad (14)$$

From eqs. VI) and X) we see that $\omega_{1d} = \nu_d = 0$, since α and β must be non trivial. Then recalling the expressions of eqs. 10 and 11, the differential operator D must be of the form

$$\begin{aligned} D(\tau) &= \omega_{1r} \wedge d\tau + \left[\left(\frac{r}{q} + t_2 \right) d\omega_{1r} + \nu_r \right] \wedge \tau \\ D(\tau) &= \omega_{1r} \wedge d\tau + r \left(\frac{1}{q} d\omega_{1r} + \omega_{2r} \right) \wedge \tau + (t_2 d\omega_{1r} + \mu_r) \wedge \tau. \end{aligned}$$

This way, we have that

$$D = \omega_{1r} \wedge \mathcal{L}_{Id} + \left(\frac{1}{q} d\omega_{1r} + \omega_{2r} \right) \wedge i_{Id} + (t_2 \lambda_{d\omega_{1r}} + \lambda_{\mu_r})$$

showing that $D \in \text{Lin}^{(q,a)}(\tilde{\Omega})$.

From c) we have

$$\begin{aligned} g \cdot (f \cdot \tau) &= g \cdot (\alpha(r, b, c) f \tau + \beta(r, b, c) df \wedge i_R \tau) = \\ &= \alpha(r, b + c, e) g \left(\alpha(r, b, c) f \tau + \beta(r, b, c) df \wedge i_R \tau \right) + \\ &\quad + \beta(r, b + c, e) dg \wedge i_R \left(\alpha(r, b, c) f \tau + \beta(r, b, c) df \wedge i_R \tau \right) = \\ &= \alpha(r, b + c, e) \alpha(r, b, c) g f \tau + \alpha(r, b + c, e) \beta(r, b, c) g df \wedge i_R \tau + \\ &\quad + \beta(r, b + c, e) \alpha(r, b, c) f dg \wedge i_R \tau + \beta(r, b + c, e) \beta(r, b, c) c f dg \wedge i_R \tau \quad (15) \end{aligned}$$

and

$$\begin{aligned} (gf) \cdot \tau &= \alpha(r, b, c + e) g f \tau + \beta(r, b, c + e) d(gf) \wedge i_R \tau = \\ &= \alpha(r, b, c + e) g f \tau + \beta(r, b, c + e) g df \wedge i_R \tau + \beta(r, b, c + e) f dg \wedge i_R \tau. \end{aligned} \quad (16)$$

Again, joining eqs. 15 and 16 as before we get the following system of equations

$$\begin{cases} \text{I) } gf \tau : & \alpha(r, b+c, e)\alpha(r, b, c) = \alpha(r, b, c+e) \\ \text{II) } g df \wedge i_{R\tau} : & \alpha(r, b+c, e)\beta(r, b, c) = \beta(r, b, c+e) \\ \text{III) } f dg \wedge i_{R\tau} : & \left(\beta(r, b+c, e)\alpha(r, b, c) + \beta(r, b+c, e)\beta(r, b, c)c \right) = \\ & = \beta(r, b, c+e) \end{cases} \quad (17)$$

which implies condition c).

For the necessity we can assume $i_{R\tau} = 0$ from where we get eq. I). Removing that equation from the system, we can choose f and g linearly independent and we are done.

Putting together eqs. 14 and 17 we get the conditions

$$\mathcal{S} : \begin{cases} \alpha(r, b, c) = \alpha(r+2, b+a, c) \\ \alpha(r, b, c) = \frac{\alpha(r, b, c+e)}{\alpha(r, b+c, e)} \\ \beta(r, b, c) = \frac{\alpha(r, b, c)}{b-a\left(\frac{r}{q} + (-1)^q t_2\right)} \end{cases}$$

The first two equations suggest a linear relation between the first two coordinates, for which the denominator of the third equation propose a formula for that. The second equation suggest a multiplicative relation between the last two coordinates.

As stated in the theorem, a formula that satisfies system \mathcal{S} can be given by

$$\alpha(r, b, c) := \frac{b-a\left(\frac{r}{q} + (-1)^q t_2\right)}{b+c-a\left(\frac{r}{q} + (-1)^q t_2\right)} \quad \beta(r, b, c) := \frac{\alpha(r, b, c)}{b-a\left(\frac{r}{q} + (-1)^q t_2\right)}$$

which clearly verifies condition a) $\alpha(-, -, 0) = 1$ of Definition 3.1.

For the case $\omega_1 = 0$, we can choose $\alpha = 1$ and $\beta = 0$ which reduces to the usual multiplication. \square

Remark 4.6. A more general formula to the one given in the previous theorem can be given by, for an appropriate t ,

$$\alpha(r, b, c) := \frac{F\left(L(b) - L\left(a\left(\frac{r}{q} + (-1)^q t\right)\right)\right)}{F\left(L(b+c) - L\left(a\left(\frac{r}{q} + (-1)^q t\right)\right)\right)} \quad \beta(r, b, c) := \frac{\alpha(r, b, c)}{b-a\left(\frac{r}{q} + (-1)^q t\right)}$$

where F is any function and L is a linear function.

5 Applications

Along this section we show two different applications of the formula for deformations that we gave in Definition 3.1. The first one shows how to linearize the usual exterior differential as we mentioned in eq. 1. The second one is related to the regularity complex of Camacho and Lins-Neto given in the introduction, see eq. 2.

5.1 Exterior differential

Definition 5.1. For $f \in S(c)$ and $\tau \in \Omega^r(b)$, we define

$$f \cdot_d \tau := \frac{b}{b+c} \left(f\tau + \frac{1}{b} df \wedge i_R \tau \right)$$

when b or c is not null and define $f \cdot_d \tau = f\tau$ when $b, c = 0$. We denote by $\Omega_d^r = \Omega^r$ and $\Omega_d = \bigoplus_{r \geq 0} \Omega_d^r$ to the \mathbb{C} -vector spaces with this action from S .

We then have:

Proposition 5.2. The exterior differential

$$\Omega_d \xrightarrow{d} \Omega_d$$

is a morphism of S -modules and the S -modules Ω_d^r are finitely generated.

Proof. It is clear that $d \in \text{Lin}^{(1,0)}(\Omega)$. Then, following Theorem 4.5, we get the formula proposed in Definition 5.1.

To see that Ω_d is finitely generated we have that $\alpha^2(r, b, 1) - \beta^2(r, b, 1) = 0$ if and only if

$$\begin{aligned} \left(\frac{b-1}{b} \right)^2 - \left(\frac{b-1}{b^2} \right)^2 &= \frac{(b-1)^2(b^2-1)}{b^4} = 0. \\ \left(\frac{b}{b+1} \right)^2 - \left(\frac{b}{(b+1)^2} \right)^2 &= \frac{b^2[(b+1)^2-1]}{(b+1)^4} = 0. \end{aligned}$$

Then, the conditions of Proposition 3.2 are verified for $b \geq 1$ and the result follows. \square

5.2 Regularity complex

In [7] we introduce a long complex $C^\bullet(\omega)$ of differential operators from the short complex of Camacho and Lins-Neto, see [1, Section III.I, p. 17] or eq. 2. In the same work we give \mathbb{C} -linear isomorphisms of $C^\bullet(\omega)$ to an S -linear complex and use it to obtain geometric information of the singular locus of the foliation defined by ω as well as its relation with first order unfoldings of ω . Here we expose a different approach to linearize the complex $C^\bullet(\omega)$.

Let us recall from [7, Section 6.1, p. 19] the following definition:

Let $\omega \in \Omega^1(a)$ such that $i_R\omega = 0$ and $\omega \wedge d\omega = 0$. We define the differential operator $\omega \triangle \in \text{Diff}^{(2,a)}(\Omega)$ as

$$\begin{array}{ccc} \Omega^r & \xrightarrow{\omega \triangle} & \Omega^{r+2} \\ \tau \mapsto & \xrightarrow{\omega \triangle} & \omega \triangle \tau := \omega \wedge d\tau + \kappa(r) d\omega \wedge \tau \end{array}$$

where $\kappa(r) := \frac{r+1}{2}$.

Using [7, Proposition 6.1.2, p. 19] we know that $\omega \triangle$ defines a differential of a complex of \mathbb{C} -vector spaces which allow us to define the graded complex $C^\bullet(\omega)$ as

$$C^\bullet(\omega) : \quad T \xrightarrow{\omega \triangle} \Omega^1 \xrightarrow{\omega \triangle} \Omega^3 \xrightarrow{\omega \triangle} \dots$$

where the 0-th differential is defined as $\omega \triangle X := \mathcal{L}_X(\omega) = i_X d\omega + di_X\omega$.

Definition 5.3. For $f \in S(c)$ and $\tau \in \Omega^r(b)$ we define the action

$$f \cdot_\Delta \tau := \frac{1}{b+c-\kappa(r)a} \left[(b-\kappa(r)a)f\tau + df \wedge i_R\tau \right]$$

and denote by $\Omega_{\omega\Delta}^r = \bigoplus_{b>\kappa(r)a} \Omega^r(b)$ and $\Omega_{\omega\Delta} = \bigoplus_{r \geq 0} \Omega_{\omega\Delta}^r$ to the \mathbb{C} -vector spaces with this action from S .

For $f \in S(c)$ and $X \in T(b)$ we also define

$$f \cdot_\Delta X = \frac{b}{b+c} fX$$

and denote by $T_{\omega\Delta} = \bigoplus_{b \geq 1} T(b)$ to the \mathbb{C} -vector space with this action from S .

We then have:

Proposition 5.4. The complex

$$C_{\omega\Delta}^\bullet(\omega) : \quad T_{\omega\Delta} \xrightarrow{\omega \triangle} \Omega_{\omega\Delta}^1 \xrightarrow{\omega \triangle} \Omega_{\omega\Delta}^3 \xrightarrow{\omega \triangle} \dots$$

is a complex of S -modules and the S -modules $\Omega_{\omega\Delta}^r$ and $T_{\omega\Delta}$ are finitely generated.

Proof. Writing $\omega \triangle$ as

$$\omega \triangle = \omega \wedge \mathcal{L}_{Id} + \frac{1}{2}d\omega \wedge i_{Id} + \frac{1}{2}\lambda_{d\omega}$$

it is clear that $\omega \triangle \in \text{Lin}^{(2,a)}(\Omega)$. Then, following Theorem 4.5, we get the formula proposed in Definition 5.3.

It is also clear that $T_{\omega \triangle}$ is finitely generated. For $\Omega_{\omega \triangle}^r$ we have that $\alpha^2(r, b, 1) - \beta^2(r, b, 1) = 0$ if and only if

$$\left(\frac{b - \kappa(r)a}{b + 1 - \kappa(r)a} \right)^2 - \left(\frac{1}{b + 1 - \kappa(r)a} \right)^2 = \frac{(b - \kappa(r)a)^2 - 1}{(b + 1 - \kappa(r)a)^2} = 0.$$

Then, the conditions of Proposition 3.2 are verified for $b > 1 + \kappa(r)a$ and the result follows. \square

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